Appendix ~~to~~ § 4.3.

Recall that (with $m(E) \le +\infty$), $f \in \mathcal{MF}^+(E)$

(*) $\displaystyle\int_E f \overset{\text{def}}{=} \sup\left\{\int_E h : h \in \mathcal{BM}_0(E),\ 0 \le h \le f \text{ on } E\right\}$

and we have established the basic properties about this integral (e.g. additivity & monotonicity etc). You should be able to see that, when $0 \le h \in \mathcal{BM}_0(E)$, the (definitions of) integral $\int_E h$ given in § 2 & in § 3 are consistent (well-defined). Since

$$\mathcal{S}^+(E) \subseteq \mathcal{MF}^+(E),$$

the definition (*) is applicable $\forall f \in \mathcal{S}^+(E)$:

$f = \sum_{j=1}^{N} b_j \chi_{B_j}$ on E (canonical representation, $B_j \subset E$ $\forall j$ etc). By additivity (of § 3), one has

$$\int_E f = \sum_{j=1}^{N} b_j\, m(B_j)$$

Since $0 < b_j$ $\forall j$ it follows that $\int_E f < +\infty$ iff each $m(B_j) < +\infty$, i.e. $f \in \mathcal{S}_0^+(E)$

(f is nonnegative and vanishes outside a set of finite measure).

Lemma 1. Let $f \in \mathcal{MF}^+(E)$. Then

$$\int_E f = \sup\left\{\int_E \varphi : \varphi \in \mathcal{S}_0^+(E),\ \varphi \leq f \text{ on } E\right\}$$

$$= \sup\left\{\int_E \varphi : \varphi \in \mathcal{S}^+(E),\ \varphi \leq f \text{ on } E\right\}$$

Pf. Let s_1, s_2 denote the above two sup respectively. Then $s_1 \leq s_2 \leq \int_E f$ by the monotonicity (of §3) and $\mathcal{S}_0^+(E) \subseteq \mathcal{S}^+(E) \subseteq \mathcal{MF}^+(E)$. Moreover let $h \in \mathcal{B}M_0(E)$ with $0 \leq h \leq f$ on E. Then

$$\int_E h \overset{\text{def}}{\underset{§2}{=}} \sup\left\{\int_E \varphi : 0 \leq \varphi \leq h \text{ on } E \text{ and } \varphi \in \mathcal{S}_0(E)\right\}$$

$$\leq s_1$$

and it follows from $(*)$ that $\int_E f \leq s_1$. QED

Note. The expressions given in Lemma 1 are in terms of materials of §1, so results of §3 can be based on §1 — without reference of §2

Monotone Convergence Th (without Fatou)

Let $0 \leq f_n \uparrow_n f$ a.e. on E with each $f_n \in \mathcal{MF}^+(E)$. (replacing E by some $E \setminus A$ with $m(A) = 0$ if nece. we assume that a.e is in fact pointwise and all function take values in $[0, +\infty]$. By the given and the monotonicity (of §3), the limit (exists in $[0,+\infty]$)

$$\lim_n \int_E f_n \leq \int_E f$$

Let $\delta \in (0,1)$ ("near" to 1). It is sufficient to show that

$$\delta \cdot \int_E f \leq \lim_n \int_E f_n$$

Let $\varphi \in \mathcal{S}_0^+(E)$, $\varphi \leq f$ on E. In view of Lemma 1, we need only to show that

$$\delta \cdot \int_E \varphi \leq \lim_n \int_E f_n \qquad (2)$$

Note that $E = \bigcup_{n \in \mathbb{N}} A_n$ and $A_n \uparrow_n$ measurable where $A_n := \{x \in E : (\delta \varphi)(x) \leq f_n(x)\}$ (pl. check two cases: $\varphi(x) = 0$ or $\varphi(x) > 0$).

A4

Since $\lambda: A \mapsto \int_A \varphi$ is a measure and the Monotone Conv. Lemma for measure tells us that

the limit on the right can be limsup or liminf

$$\int_E \delta\varphi = \lim_n \int_{A_n} (\delta\varphi) \leq \overline{\lim_n} \int_{A_n} f_n \leq \lim_n \int_E f_n$$

$\delta\varphi \leq f_n$ on A_n

so (2) is shown.

Fatou's Lemma $0 \leq f_n \in MF^+(E)$ and $\lim_n f_n(x) = f(x)$ a.e x in E. Then

$$\int_E f \leq \liminf_n \int_E f_n .$$

Pf. ~~A~~ again, we use ptwise instead of a.e.

Let $g_N = \bigwedge_{n \geq N} f_n$ $(N \in \mathbb{N})$ be defined ptwisely:

$$g_N = \inf\{f_n(x) : n \in \mathbb{N}, n \geq N\}.$$

Then each $g_N \in MF^+(E)$ and $g_N \uparrow_N f$ with $f = \bigvee_{N \in \mathbb{N}} g_N$. Then, by MCTh,

Same as "liminf"

$$\int_E f = \lim_N \int g_N = \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \int g_n \leq \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} \int f_n$$

$$= \liminf_n \int f_n .$$